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Trefoil Symmetry IV: Basic Enhanced Superspace for the Minimal Vector Clover Extension

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We construct a differential representation and covariant derivatives of the minimal vector clover extension of the Poincaré algebra. In analogous way as in the supersymmetric case, there arises an enhanced superspace which allows to define superfields. The action of group transformations on such superfields determines a representation out of which the covariant derivatives are obtained.

KEY WORDS: supersymmetry; superspace; graded symmetries; noncommutative field theory; representation theory.

1. INTRODUCTION

The novel extensions of the Poincaré algebra proposed by the so-called *trefoil* symmetries in literature (Wills-Toro, 2001a,b; Wills-Toro *et al.*, 2001) provide a promising approach to tackle very old problems in quantum field theory (QFT). The trefoil symmetries generalize the mathematical structures that lead to supersymmetry while maintaining the main portion of the hypothesis of the Coleman–Mandula and Haag–Łopuszanski–Sohnius no-go theorems (Coleman and Mandula, 1967; Haag *et al.*, 1975). The trefoil symmetries involving only $\mathbb{Z}_4 \times \mathbb{Z}_4$ graded symmetry generators and only multiplets of generators of integer spin have been called *clover* extensions.

A minimal nontrivial clover extension involving only vector multiplets of symmetry generators was obtained in literature (Will-Toro, 2001a) that has been called the *minimal vector clover extension*. For a brief review see Appendix A. Using this symmetry, the aim of the present paper is to develop the superspace formalism—Section 2, a suited algebra of graded differential operators—Section 3, the determination of a representation of such symmetry generators—Section 4, as

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well as the corresponding representation for its covariant derivatives—Section 5. After the concluding remarks we include the algebra *minimal vector clover extension* and its structure constants—Appendix A, some useful formulae on the metrics and the arrays of structure constants—Appendix B, and some algebraic relations among covariant derivatives—Appendix C. The later will give the reader some tools employed for concrete model building in the forthcoming contributions of this paper series.

2. ENHANCED SUPERSPACE

The superspace formalism constitutes a very powerful tool for the construction of models in supersymmetry (Ferrara *et al.*, 1974; Ferrara and Zumino, 1975; Salam and Strathdee, 1974). We develop here a superspace formalism for the minimal vector clover extension introduced in Wills-Toro (2001a), whose $(\mathbb{Z}_4 \times \mathbb{Z}_4; q)$-graded Lie algebra is given in Appendix A.

We want to represent the algebra on superfields Φ, so that for each symmetry generator $\mathcal{O}$ we obtain a differential operator $\delta_{\mathcal{O}}$ that reproduces the action of the operator on a superfield Φ:

$$\mathrm{i}[\![\mathcal{O}, \Phi]\!] = \delta_{\mathcal{O}}\Phi; \quad \forall\mathcal{O} \in \text{algebra}. \tag{2.1}$$

From the algebraic relation

$$[\![\mathcal{O}, \mathcal{O}']\!] = \mathrm{i}\mathcal{O}'', \tag{2.2}$$

assuming that the trivial index is assigned to Φ (i.e. $S_t(\Phi) = \tilde{o} = (0, 0) \in \mathbb{Z}_4 \times \mathbb{Z}_4$) and using the graded Jacobi associativity, we obtain

$$\mathrm{i}[\![-\mathrm{i}[\![\mathcal{O}, \mathcal{O}']\!], \Phi]\!] = [\![\delta_{\mathcal{O}}, \delta_{\mathcal{O}'}]\!]\Phi. \tag{2.3}$$

Accordingly

$$[\![\delta_{\mathcal{O}}, \delta_{\mathcal{O}'}]\!] = \delta_{\mathcal{O}''}. \tag{2.4}$$

To obtain the transformations $\delta_{\mathcal{O}}$, $\mathcal{O} \in$ algebra, we study first transformations involving nonvanishing parameters only for the generators $P_{(0)\mu}$, $T_{(i)r}$, $\overline{T}_{(i)\dot{r}}$, $P_{(i)\mu}$. This subset of generators builds a nilpotent subalgebra as can be inferred from the q-commutator relations given in Appendix A: at most triple q-commutators can be nonvanishing. We *assume* from now on that monomials involving identical subindex and superindex pairs have a summation over all allowed values for such an index. This rule disregards minus signs that are only relevant for grading index assignments. For a brief review of the grading group and its notation see Appendix A.

A (finite) symmetry transformation involving the mentioned subalgebra will have the form

$$G(\chi, \xi, \overline{\xi}, \beta) = e^{\mathrm{i}\chi_{(0)}^{-\mu} P_{(0)\mu} + \sum_{j=1,2,3} \left(\mathrm{i}\xi_{(j)}^{-r} T_{(j)r} + \mathrm{i}\overline{T}_{(j)\dot{r}} \overline{\xi}_{(j)}^{-\dot{r}} + \mathrm{i}\beta_{(j)}^{-\mu} P_{(j)\mu}\right)}. \tag{2.5}$$

The parameters $\chi^{-\mu}_{(0)}$, $\xi^{-r}_{(j)}$, $\bar{\xi}^{-\dot{r}}_{(j)}$, $\beta^{-\mu}_{(j)}$ carry, respectively, the grading indices $-(0)\mu$, $-(i)r$, $-(i)\dot{r}$, $-(i)\mu$ so that they provide products in (2.5) carrying neutral index, and thus building a Lie group element (Wills-Toro, 1997, 2001a). Notice that in order to emphasize the different roles played by parameters and symmetry generators, the class component indices of the parameters are denoted to the left of the class indices, and they carry the overall sign of its group element.

We consider now the products of two group elements. For that we recall the Baker–Campbell–Hausdorff formula

$$e^A e^B = e^{A+B+(1/2)[A,B]+(1/12)[A,[A,B]]-(1/12)[B,[A,B]]+\cdots}. \tag{2.6}$$

Since there are no nontrivial q-commutators of order higher than three for the chosen generators, all the relevant terms are those explicitly given in (2.6). Recalling that the connection among the Lie algebra and the graded Lie algebra (Wills-Toro, 1997, 2001a) is given by

$$\left[\omega^{-s}_{(\mu)}\mathcal{O}_{(\mu)s},\, \omega'^{-t}_{(\nu)}\mathcal{O}'_{(\nu)t}\right] = \omega'^{-t}_{(\nu)}\omega^{-s}_{(\mu)}[\![\mathcal{O}_{(\mu)s},\, \mathcal{O}'_{(\nu)t}]\!], \tag{2.7}$$

we obtain

$$G(a,\rho,\bar{\rho},\alpha)G(\chi,\xi,\bar{\xi},\beta) = G(\hat{\chi},\hat{\xi},\hat{\bar{\xi}},\hat{\beta}), \tag{2.8}$$

where, for $i, j, k \in \{1, 2, 3\}$ we have

$$\hat{\beta}^{-\mu}_{(i)} = \beta^{-\mu}_{(i)} + \alpha^{-\mu}_{(i)}, \tag{2.9}$$

$$\hat{\xi}^{-r}_{(i)} = \xi^{-r}_{(i)} + \rho^{-r}_{(i)} + \frac{\mathrm{i}}{2}\sum_{k\neq i}\beta^{-\rho}_{(k\dagger i)}\alpha^{-\sigma}_{(k)}\eta^{r}(k, k\dagger i)_{\sigma\rho}, \tag{2.10}$$

$$\hat{\bar{\xi}}^{-\dot{r}}_{(i)} = \bar{\xi}^{-\dot{r}}_{(i)} + \bar{\rho}^{-\dot{r}}_{(i)} + \frac{\mathrm{i}}{2}\sum_{k\neq i}\beta^{-\rho}_{(k\dagger i)}\alpha^{-\sigma}_{(k)}\hat{\eta}^{\dot{r}}(k, k\dagger i)_{\sigma\rho}, \tag{2.11}$$

$$\begin{aligned}
\hat{\chi}^{-\mu}_{(0)} = \chi^{-\mu}_{(0)} + a^{-\mu}_{(0)} + \sum_{i}\Bigg(&\Bigg\{\frac{\mathrm{i}}{2}\left(\beta^{-\nu}_{(i)}\rho^{-r}_{(i)} - \alpha^{-\nu}_{(i)}\xi^{-r}_{(r)}\right) \\
&- \frac{1}{12}\left(\beta^{-\nu}_{(i)} - \alpha^{-\nu}_{(i)}\right)\sum_{k\neq i}\beta^{-\rho}_{(k\dagger i)}\alpha^{-\sigma}_{(k)}\eta^{r}(k, k\dagger i)_{\sigma\rho}\Bigg\}K_r(i)_\nu{}^{\mu} \\
&+ \Bigg\{\frac{\mathrm{i}}{2}\left(\beta^{-\nu}_{(i)}\bar{\rho}^{-\dot{r}}_{(i)} - \alpha^{-\nu}_{(i)}\bar{\xi}^{-\dot{r}}_{(i)}\right) \\
&- \frac{1}{12}\left(\beta^{-\nu}_{(i)} - \alpha^{-\nu}_{(i)}\right)\sum_{k\neq i}\beta^{-\rho}_{(k\dagger i)}\alpha^{-\sigma}_{(k)}\hat{\eta}^{\dot{r}}(k, k\dagger i)_{\sigma\rho}\Bigg\}\hat{K}_{\dot{r}}(i)_\nu{}^{\mu}\Bigg),
\end{aligned} \tag{2.12}$$

The representation in terms of differential operators requires the formal definition of derivation with respect to graded parameters. The following section is devoted to this aim.

3. GRADED DIFFERENTIAL OPERATORS

The index assignment for the parameters is given by the index-assigning function S_t:

$$
\begin{aligned}
S_t\left(\chi_{(0)}^{-\mu}\right) &= S_t\left(\chi_{-\mu(0)}\right) = -(0)\mu = (0)\mu, \\
S_t\left(\xi_{(i)}^{-r}\right) &= S_t\left(\xi_{-r(i)}\right) = -(i)r, \\
S_t\left(\bar{\xi}_{(i)}^{-\dot{r}}\right) &= S_t\left(\bar{\xi}_{-\dot{r}(i)}\right) = -(i)\dot{r}, \\
S_t\left(\beta_{(i)}^{-\sigma}\right) &= S_t\left(\beta_{-\sigma(i)}\right) = -(i)\sigma,
\end{aligned}
\tag{3.1}
$$

where the relation among parameter (variables) with upper and lower indices is given by

$$
\begin{aligned}
\chi_{-\mu(0)} &= \epsilon^P(0)_{\mu\nu}\chi_{(0)}^{-\nu}, \\
\chi_{-r(i)} &= \epsilon^T(i)_{rs}\chi_{(i)}^{-s}, \\
\bar{\xi}_{-\dot{r}(i)} &= \bar{\epsilon}^T(i)_{\dot{r}\dot{s}}\bar{\xi}_{(i)}^{-\dot{s}}, \\
\beta_{-\sigma(i)} &= \epsilon^P(i)_{\sigma\rho}\beta_{(i)}^{-\rho}.
\end{aligned}
\tag{3.2}
$$

The invariant metrics and some useful relations among the structure constant arrays are given in Appendix B.

We introduce formal differential operators following (Wills-Toro, 1994), with index assignment given by

$$
\begin{aligned}
S_t\left(\partial_{\chi_{(0)}^{-\mu}}\right) &= S_t\left(\partial_{\chi_{-\mu(0)}}\right) = (0)\mu = -(0)\mu, \\
S_t\left(\partial_{\xi_{(i)}^{-r}}\right) &= S_t\left(\partial_{\xi_{-r(i)}}\right) = (i)r, \\
S_t\left(\partial_{\bar{\xi}_{(i)}^{-\dot{r}}}\right) &= S_t\left(\partial_{\bar{\xi}_{-\dot{r}(i)}}\right) = (i)\dot{r}, \\
S_t\left(\partial_{\beta_{(i)}^{-\sigma}}\right) &= S_t\left(\partial_{\beta_{-\sigma(i)}}\right) = (i)\sigma.
\end{aligned}
\tag{3.3}
$$

The defining relations for the action of differential operators on graded parameter variables is given as proposed in Wills-Toro (1994). They are just the articulation of a q-Leibnitz rule for the derivation with respect to graded parameters:

$$
\begin{aligned}
[\![\partial_{\chi_{(0)}^{-\mu}}, \chi_{(0)}^{-\nu}]\!] &= \delta_\mu^\nu, & [\![\partial_{\chi_{-\mu(0)}}, \chi_{-\nu(0)}]\!] &= \delta_\nu^\mu, \\
[\![\partial_{\xi_{(i)}^{-r}}, \xi_{(i)}^{-s}]\!] &= \delta_r^s, & [\![\partial_{\xi_{-r(i)}}, \xi_{-s(i)}]\!] &= \delta_s^r, \\
[\![\partial_{\bar{\xi}_{(i)}^{-\dot{r}}}, \bar{\xi}_{(i)}^{-\dot{s}}]\!] &= \delta_{\dot{r}}^{\dot{s}}, & [\![\partial_{\bar{\xi}_{-\dot{r}(i)}}, \bar{\xi}_{-\dot{s}(i)}]\!] &= \delta_{\dot{s}}^{\dot{r}}, \\
[\![\partial_{\beta_{(i)}^{-\sigma}}, \beta_{(i)}^{-\rho}]\!] &= \delta_\sigma^\rho, & [\![\partial_{\beta_{-\sigma(i)}}, \beta_{-\rho(i)}]\!] &= \delta_\rho^\sigma.
\end{aligned}
\tag{3.4}
$$

All further q-commutator combinations of such differential operators and parameters vanish with the exception of

$$
\begin{aligned}
\left[\!\left[\partial_{\chi_{(0)}^{-\mu}}, \chi_{-\nu(0)}\right]\!\right] &= \epsilon^{P}(0)_{\nu\mu}, \qquad \left[\!\left[\partial_{\chi_{-\mu(0)}}, \chi_{(0)}^{-\nu}\right]\!\right] = \epsilon^{P}(0)^{\nu\mu},\\
\left[\!\left[\partial_{\xi_{(i)}^{-r}}, \xi_{-t(i)}\right]\!\right] &= \epsilon^{T}(i)_{tr}, \qquad \left[\!\left[\partial_{\xi_{-r(i)}}, \xi_{(i)}^{-t}\right]\!\right] = \epsilon^{T}(i)^{tr},\\
\left[\!\left[\partial_{\bar{\xi}_{(i)}^{-\dot{r}}}, \bar{\xi}_{-\dot{t}(i)}\right]\!\right] &= \bar{\epsilon}^{T}(i)_{\dot{t}\dot{r}}, \qquad \left[\!\left[\partial_{\bar{\xi}_{-\dot{r}(i)}}, \bar{\xi}_{(i)}^{-\dot{t}}\right]\!\right] = \bar{\epsilon}^{T}(i)^{\dot{t}\dot{r}},\\
\left[\!\left[\partial_{\beta_{(i)}^{-\sigma}}, \beta_{-\rho(i)}\right]\!\right] &= \epsilon^{P}(i)_{\rho\sigma}, \qquad \left[\!\left[\partial_{\beta_{-\sigma(i)}}, \beta_{(i)}^{-\rho}\right]\!\right] = \epsilon^{P}(i)^{\rho\sigma},
\end{aligned}
\tag{3.5}
$$

which follow from (3.2) to (3.4). When the reader becomes familiar with the properties of graded operators, he/she might replace $\partial_{\chi_{(0)}^{-\mu}}$ by $\partial_{(0)\mu}$, $\partial_{\chi_{-\mu(0)}}$ by $\partial_{(0)}^{\mu}$ and so on.

From the q-commutator property

$$
[\![A_a, B_b C_c]\!] = [\![A_a, B_b]\!] C_c + q_{a,b} B_b [\![A_a, C_c]\!], \tag{3.6}
$$

the action of derivatives on products is easily inferred.

4. BASIC REPRESENTATION OF THE GENERATORS

From the transformation of parameters $(\chi, \xi, \bar{\xi}, \beta) \mapsto (\hat{\chi}, \hat{\xi}, \hat{\bar{\xi}}, \hat{\beta})$ we can obtain a representation of the generators in terms of graded differential operators. This representation will be called the *basic representation*. It can be considered a real representation since it handles the ξ and $\bar{\xi}$ variables on the same footing.

$$
\begin{aligned}
\delta_{P_{(0)\mu}} &= \partial_{\chi_{(0)}^{-\mu}},\\
\delta_{T_{(i)r}} &= \partial_{\xi_{(i)}^{-r}} - \frac{\mathrm{i}}{2}\beta_{(i)}^{-\nu}\hat{K}_r^*(i)_\nu{}^\mu \partial_{\chi_{(0)}^{-\mu}},\\
\delta_{\bar{T}_{(i)\dot{r}}} &= \partial_{\bar{\xi}_{(i)}^{-\dot{r}}} - \frac{\mathrm{i}}{2}\beta_{(i)}^{-\nu}K_{\dot{r}}^*(i)_\nu{}^\mu \partial_{\chi_{(0)}^{-\mu}},\\
\delta_{P_{(k)\sigma}} &= \partial_{\beta_{(k)}^{-\sigma}} - \frac{\mathrm{i}}{2}\sum_{i\neq k}\beta_{(k\dagger i)}^{-\rho}\left(\eta^r(k\dagger i,k)_{\rho\sigma}\partial_{\xi_{(i)}^{-r}} + \hat{\eta}^{\dot{r}}(k\dagger i,k)_{\rho\sigma}\partial_{\bar{\xi}_{(i)}^{-\dot{r}}}\right)\\
&\quad - \frac{\mathrm{i}}{2}\left(\xi_{(k)}^{-r}K_r(k)_\sigma{}^\mu + \bar{\xi}_{(k)}^{-\dot{r}}\hat{K}_{\dot{r}}(k)_\sigma{}^\mu\right)\partial_{\chi_{(0)}^{-\mu}}\\
&\quad - \frac{1}{12}\sum_{i\neq k}\beta_{(k\dagger i)}^{-\rho}\beta_{(i)}^{-\nu}\left\{\eta^r(k\dagger i,k)_{\rho\sigma}\hat{K}_r^*(i)_\nu{}^\mu + \hat{\eta}^{\dot{r}}(k\dagger i,k)_{\rho\sigma}K_{\dot{r}}^*(i)_\nu{}^\mu\right\}\partial_{\chi_{(0)}^{-\mu}}.
\end{aligned}
\tag{4.1}
$$

Observe that we write $\hat{K}_r^*(i)_\nu{}^\mu$ with undotted r index and $K_{\dot{r}}^*(i)_\nu{}^\mu$ with dotted $\dot{r}$ index as it follows after applying involution to the corresponding q-commutation relations. Strictly speaking, we should write $\delta_{\mathcal{O}}^{\Phi}$ instead of $\delta_{\mathcal{O}}$ in the equations above. But the differential representation considered here refers to the same fixed superfield Φ, so we will use easy notation, and skip this reference.

It is a long, but straightforward calculation to verify that this differential representation (4.1) fulfills the relations (2.4) for the corresponding ones in (2.2) for the minimal vector clover extension:

$$[\![\delta_{P_{(k)\sigma}}, \delta_{P_{(j)\alpha}}]\!] = -\mathrm{i}\big(\eta^r(k, j)_{\sigma\alpha}\, \delta_{T_{(k\dagger j)r}} + \hat{\eta}^{\dot{r}}(k, j)_{\sigma\alpha}\delta_{\bar{T}_{(k\dagger j)\dot{r}}}\big), \quad \text{for } k \neq j, \tag{4.2}$$

$$[\![\delta_{T_{(j)s}}, \delta_{P_{(k)\sigma}}]\!] = -\mathrm{i}\delta_{jk} K_s(k)_\sigma{}^\mu \delta_{P_{(0)\mu}}, \tag{4.3}$$

$$[\![\delta_{\bar{T}_{(j)\dot{s}}}, \delta_{P_{(k)\sigma}}]\!] = -\mathrm{i}\delta_{jk} \hat{K}_{\dot{s}}(k)_\sigma{}^\mu \delta_{P_{(0)\mu}}, \tag{4.4}$$

and all the further q-commutators between these differential operators vanish.

There remain to determine the differential operators associated to the Lorentz generators. The corresponding operators can be inferred using the close analogy with the supersymmetric case:

$$\begin{aligned}\delta_{T(0)_i} &= q_{(0)\nu,(0)i}\, \chi_{(0)}^{-\nu} \sigma^P(i, 0)_\nu{}^\rho \partial_{\chi_{(0)}^{-\rho}} + q_{(j)r,(0)i} \xi_{(j)}^{-r} \sigma^T(i, j)_r{}^s \partial_{\xi_{(j)}^{-s}} \\ &\quad + q_{(j)\nu,(0)i} \beta_{(j)}^{-\nu} \sigma^P(i, j)_\nu{}^\rho \partial_{\beta_{(j)}^{-\rho}},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\delta_{\bar{T}_{(0)i}} &= \chi_{(0)}^{-\nu} \bar{\sigma}^P(i, 0)^\rho{}_\nu \partial_{\chi_{(0)}^{-\rho}} + \bar{\xi}^{-\dot{r}}(j) \bar{\sigma}^{\bar{T}}(i, j)^{\dot{s}}{}_{\dot{r}} \partial_{\xi^{-\dot{s}}(j)} \\ &\quad + \beta_{(j)}^{-\nu} \bar{\sigma}^P(i, j)^\rho{}_\nu \partial_{\beta_{(j)}^{-\rho}}.\end{aligned} \tag{4.6}$$

The connection with the standard form used for presenting Lorentz generators is given by the formula:

$$M^{\alpha\beta} = \mathrm{i}\epsilon_\nu^{\mu\,\alpha\beta}\big(\sigma^P(i, 0)_\mu{}^\nu T_{(0)i} - \bar{\sigma}^P(i, 0)^\nu{}_\mu \bar{T}_{(0)i}\big), \tag{4.7}$$

where $\epsilon^{\theta\beta\delta\tau} = -\epsilon_{\theta\beta\delta\tau}$ is the totally antisymmetric Levi–Civita tensor: $\epsilon_{0123} = -\epsilon^{0123} = 1$. The Levi–Civita tensor can, in turn, be expressed in terms of the symmetric vector metric and Lorentz representation by

$$\begin{aligned}\epsilon^{\theta\beta\delta\tau} &= \mathrm{i}\big(\epsilon^P(0)^{\beta\mu}\bar{\sigma}^P(i, 0)^\theta{}_\mu - \epsilon^P(0)^{\theta\mu}\bar{\sigma}^P(i, 0)^\beta{}_\mu\big)\, \epsilon^P(0)^{\tau\xi}\bar{\sigma}^P(i, 0)^\delta{}_\xi \\ &\quad - \mathrm{i}\big(\epsilon^P(0)^{\beta\mu}\sigma^P(i, 0)_\mu{}^\theta - \epsilon^P(0)^{\theta\mu}\sigma^P(i, 0)_\mu{}^\beta\big)\, \epsilon^P(0)^{\tau\xi}\sigma^P(i, 0)_\xi{}^\delta.\end{aligned} \tag{4.8}$$

We can imagine that there might be a Levi–Civita tensor associated to the novel classes. We will address this and further questions in a forthcoming contribution of this paper series on Casimir operators.

5. COVARIANT DERIVATIVES BASIC REPRESENTATION

Once the differential relation for the minimal vector clover extension has been successfully constructed, there appears naturally the question about the existence of covariant derivatives. These might be gained from the action of the differential operators from the right (Ferrara *et al.*, 1974). The covariant derivatives are differential operators that q-commute with all the novel generators of the minimal vector clover extension and with the generators of the space–time translations. We define the covariant derivative in close analogy with the supersymmetry case:

$$D_{T_{(i)r}} = \partial_{\xi^{-r}_{(i)}} + \frac{\mathrm{i}}{2}\beta^{-\nu}_{(i)}\hat{K}^{*}_{r}(i)_{\nu}{}^{\mu}\partial_{\chi^{-\mu}_{(0)}}, \tag{5.1}$$

$$D_{\bar{T}_{(i)\dot{r}}} = \partial_{\bar{\xi}^{-\dot{r}}_{(i)}} + \frac{\mathrm{i}}{2}\beta^{-\nu}_{(i)}K^{*}_{\dot{r}}(i)_{\nu}{}^{\mu}\partial_{\chi^{-\mu}_{(0)}}, \tag{5.2}$$

$$\begin{aligned} D_{P_{(k)\sigma}} &= \partial_{\beta^{-\sigma}_{(k)}} + \frac{\mathrm{i}}{2}\sum_{i\neq k}\beta^{-\rho}_{(k\dagger i)}\left\{\eta^{r}(k\dagger i,k)_{\rho\sigma}\partial_{\xi^{-r}_{(i)}} + \hat{\eta}^{\dot{r}}(k\dagger i,k)_{\rho\sigma}\partial_{\bar{\xi}^{-\dot{r}}_{(i)}}\right\} \\ &\quad + \frac{\mathrm{i}}{2}\left(\xi^{-r}_{(k)}K_{r}(k)_{\sigma}{}^{\mu} + \bar{\xi}^{-\dot{r}}_{(k)}\hat{K}_{\dot{r}}(k)_{\sigma}{}^{\mu}\right)\partial_{\chi^{-\mu}_{(0)}} \\ &\quad - \frac{1}{12}\sum_{i\neq k}\beta^{-\rho}_{(k\dagger i)}\beta^{-\nu}_{(i)}\left\{\eta^{r}(k\dagger i,k)_{\rho\sigma}\hat{K}^{*}_{r}(i)_{\nu}{}^{\mu}\right. \\ &\quad \left. + \hat{\eta}^{\dot{r}}(k\dagger i,k)_{\rho\sigma}K^{*}_{\dot{r}}(i)_{\nu}{}^{\mu}\right\}\partial_{\chi^{-\mu}_{(0)}}. \end{aligned} \tag{5.3}$$

With these definitions we verify

$$[\![D_{T_{(j)s}}, \delta_{P_{(k)\sigma}}]\!] = 0, \qquad [\![D_{\bar{T}_{(i)\dot{s}}}, \delta_{P_{(k)\sigma}}]\!] = 0, \tag{5.4}$$

$$[\![D_{T_{(j)s}}, \delta_{T_{(i)r}}]\!] = 0, \qquad [\![D_{\bar{T}_{(i)\dot{s}}}, \delta_{T_{(i)r}}]\!] = 0, \tag{5.5}$$

$$[\![D_{T_{(j)s}}, \delta_{\bar{T}_{(i)\dot{r}}}]\!] = 0, \qquad [\![D_{\bar{T}_{(i)\dot{s}}}, \delta_{\bar{T}_{(i)\dot{r}}}]\!] = 0, \tag{5.6}$$

$$[\![D_{T_{(j)s}}, \delta_{P_{(0)\mu}}]\!] = 0, \qquad [\![D_{\bar{T}_{(i)\dot{s}}}, \delta_{P_{(0)\mu}}]\!] = 0, \tag{5.7}$$

$$[\![D_{P_{(k)\sigma}}, \delta_{P_{(j)\alpha}}]\!] = 0, \qquad [\![D_{P_{(k)\sigma}}, \delta_{P_{(0)\mu}}]\!] = 0, \tag{5.8}$$

$$[\![D_{P_{(k)\sigma}}, \delta_{T_{(j)r}}]\!] = 0, \qquad [\![D_{P_{(k)\sigma}}, \delta_{\bar{T}_{(i)\dot{r}}}]\!] = 0. \tag{5.9}$$

The covariant derivatives among them fulfill the following q-commutation relations:

$$\begin{aligned} [\![D_{P_{(k)\sigma}}, D_{P_{(j)\alpha}}]\!] &= \mathrm{i}\big(\eta^{r}(k,j)_{\sigma\alpha}\,D_{T_{(k\dagger j)r}} \\ &\qquad + \hat{\eta}^{\dot{r}}(k,j)_{\sigma\alpha}\,D_{\bar{T}_{(k\dagger j)\dot{r}}}\big), \quad \text{for}\, k\neq j, \end{aligned} \tag{5.10}$$

$$[\![D_{T_{(j)s}}, D_{P_{(k)\sigma}}]\!] = \mathrm{i}\delta_{jk}\,K_{s}(k)_{\sigma}{}^{\mu}\,\partial_{\chi^{-\mu}_{(0)}}, \tag{5.11}$$

$$[\![D_{\bar{T}_{(j)\dot{s}}}, D_{P_{(k)\sigma}}]\!] = \mathrm{i}\delta_{jk}\,\hat{K}_{\dot{s}}(k)_{\sigma}{}^{\mu}\,\partial_{\chi^{-\mu}_{(0)}}, \tag{5.12}$$

and all the further q-commutators among covariant derivatives vanish. We observe that the main features of the supersymmetry case are maintained, namely, the covariant derivative $D_{\mathcal{O}}$ is obtained changing the sign of the explicit imaginary unit in the representation $\delta_{\mathcal{O}}$.

The operator $\delta_{\mathcal{O}}$ fulfills an algebra very close to the minimal vector clover extension algebra up to a factor $-\mathrm{i}$ in the result of the q-commutators, as expressed in (2.2)–(2.4). The covariant derivatives $D_{\mathcal{O}}$ fulfill algebraic relations which coincide with those of the minimal vector clover extension up to a factor $+\mathrm{i}$ in the result of the q-commutators. Actually, a whole set of algebraic relations is obtained in analogous way as done in the supersymmetry case. Some useful relations relevant for model building, construction of projector operators, etc. are given in Appendix C.

6. CONCLUSIONS

One of the basic challenges of any symmetry in QFT is its realizability in concrete sensible models. This exploration points the convenience or inconvenience of certain options. The trefoil symmetries have been developed having in mind very concrete applications. The construction of such concrete models for the minimal vector clover extension is made transparent using the powerful methods of superspace formalism. The construction of concrete representations arises naturally as well as the corresponding covariant derivatives. All the features observed in the supersymmetric superspace formalism are obtained for the minimal vector clover extension case. The reader could wander about the existence of further representations, as the chiral and antichiral representations of supersymmetry. In the next contribution of this paper series, we determine novel representations as well as the relations among the diverse representations.

APPENDIX A: THE MINIMAL VECTOR CLOVER EXTENSION

The *trefoil symmetries* are graded Lie algebraic extensions with involution of the Poincaré algebra (Wills-Toro, 2001a,b; Wills-Toro *et al.*, 2001). These extensions respect slightly more generalized hypothesis than the ones underlying the no-go theorems in Coleman and Mandula (1967) and Haag *et al.* (1975). Here, the Poincaré generators do not always act through commutators on further symmetry generators, and more generally, the group structure has an underlying (I, q)-graded Lie algebraic structure with involution. For a basic introduction to these algebras and group gradings see Wills-Toro (1995, 1997, 2001a).

The trefoil symmetries certainly include supersymmetry, which is a $\mathbb{Z}_2$-graded extension of the Poincaré algebra. The trefoil symmetries which involve only $\mathbb{Z}_4 \times \mathbb{Z}_4$-graded parameters and novel generators of integer spin have been called *clover extensions* (Wills-Toro, 2001a,b; Wills-Toro *et al.*, 2001). We call the *minimal vector clover extension* a $(\mathbb{Z}_4 \times \mathbb{Z}_4; q)$-graded Lie algebraic extension of the Poincaré algebra which is minimal (no subalgebra extends nontrivially the Poincaré algebra) and whose novel generators build symmetric and antisymmetric vectors. In this extension a space–time translation is obtained through the composition of three symmetric vectors.

Table AI. Addition Table of the Group $\mathbb{Z}_4 \times \mathbb{Z}_4$

+	(0)0	(0)1	(0)2	(0)3	(1)0	(1)1	(1)2	(1)3	(2)0	(2)1	(2)2	(2)3	(3)0	(3)1	(3)2	(3)3
(0,0) ≡ (0)0	(0)0	(0)1	(0)2	(0)3	(1)0	(1)1	(1)2	(1)3	(2)0	(2)1	(2)2	(2)3	(3)0	(3)1	(3)2	(3)3
(2,0) ≡ (0)1	(0)1	(0)0	(0)3	(0)2	(1)1	(1)0	(1)3	(1)2	(2)3	(2)2	(2)1	(2)0	(3)2	(3)3	(3)0	(3)1
(0,2) ≡ (0)2	(0)2	(0)3	(0)0	(0)1	(1)2	(1)3	(1)0	(1)1	(2)1	(2)0	(2)3	(2)2	(3)3	(3)2	(3)1	(3)0
(2,2) ≡ (0)3	(0)3	(0)2	(0)1	(0)0	(1)3	(1)2	(1)1	(1)0	(2)2	(2)3	(2)0	(2)1	(3)1	(3)0	(3)3	(3)2
(1,0) ≡ (1)0	(1)0	(1)1	(1)2	(1)3	(0)1	(0)0	(0)3	(0)2	(3)1	(3)2	(3)0	(3)3	(2)1	(2)3	(2)2	(2)0
(3,0) ≡ (1)1	(1)1	(1)0	(1)3	(1)2	(0)0	(0)1	(0)2	(0)3	(3)3	(3)0	(3)2	(3)1	(2)2	(2)0	(2)1	(2)3
(1,2) ≡ (1)2	(1)2	(1)3	(1)0	(1)1	(0)3	(0)2	(0)1	(0)0	(3)2	(3)1	(3)3	(3)0	(2)0	(2)2	(2)3	(2)1
(3,2) ≡ (1)3	(1)3	(1)2	(1)1	(1)0	(0)2	(0)3	(0)0	(0)1	(3)0	(3)3	(3)1	(3)2	(2)3	(2)1	(2)0	(2)2
(0,1) ≡ (2)0	(2)0	(2)3	(2)1	(2)2	(3)1	(3)3	(3)2	(3)0	(0)2	(0)0	(0)1	(0)3	(1)1	(1)2	(1)0	(1)3
(0,3) ≡ (2)1	(2)1	(2)2	(2)0	(2)3	(3)2	(3)0	(3)1	(3)3	(0)0	(0)2	(0)3	(0)1	(1)3	(1)0	(1)2	(1)1
(2,3) ≡ (2)2	(2)2	(2)1	(2)3	(2)0	(3)0	(3)2	(3)3	(3)1	(0)1	(0)3	(0)2	(0)0	(1)2	(1)1	(1)3	(1)0
(2,1) ≡ (2)3	(2)3	(2)0	(2)2	(2)1	(3)3	(3)1	(3)0	(3)2	(0)3	(0)1	(0)0	(0)2	(1)0	(1)3	(1)1	(1)2
(3,3) ≡ (3)0	(3)0	(3)2	(3)3	(3)1	(2)1	(2)2	(2)0	(2)3	(1)1	(1)3	(1)2	(1)0	(0)3	(0)0	(0)2	(0)1
(1,1) ≡ (3)1	(3)1	(3)3	(3)2	(3)0	(2)3	(2)0	(2)2	(2)1	(1)2	(1)0	(1)1	(1)3	(0)0	(0)3	(0)1	(0)2
(1,3) ≡ (3)2	(3)2	(3)0	(3)1	(3)3	(2)2	(2)1	(2)3	(2)0	(1)0	(1)2	(1)3	(1)1	(0)2	(0)1	(0)3	(0)0
(3,1) ≡ (3)3	(3)3	(3)1	(3)0	(3)2	(2)0	(2)3	(2)1	(2)2	(1)3	(1)1	(1)0	(1)2	(0)1	(0)2	(0)0	(0)3

The grading group $\mathbb{Z}_4 \times \mathbb{Z}_4$ has the addition given by Table AI. The conversion between the standard notation on the group elements (as couples (n, m) with $n, m \in \{0, 1, 2, 3\}$) and the notation used in this application is also given there. The q-function is given in Table AII.

Table AII. q-Function for $\mathbb{Z}_4 \times \mathbb{Z}_4$

$q^{\mathbb{Z}_4 \times \mathbb{Z}_4}$	(0)0	(0)1	(0)2	(0)3	(1)0	(1)1	(1)2	(1)3	(2)0	(2)1	(2)2	(2)3	(3)0	(3)1	(3)2	(3)3
(0,0) ≡ (0)0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
(2,0) ≡ (0)1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1
(0,2) ≡(0)2	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1
(2,2) ≡ (0)3	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1
(1,0) ≡ (1)0	1	1	−1	−1	1	1	−1	−1	i	−i	−i	i	−i	i	−i	i
(3,0) ≡ (1)1	1	1	−1	−1	1	1	−1	−1	−i	i	i	−i	i	−i	i	−i
(1,2) ≡(1)2	1	1	−1	−1	−1	−1	1	1	i	−i	−i	i	i	−i	i	−i
(3,2) ≡ (1)3	1	1	−1	−1	−1	−1	1	1	−i	i	i	−i	−i	i	−i	i
(0,1) ≡ (2)0	1	−1	1	−1	−i	i	−i	i	1	1	−1	−1	i	−i	−i	i
(0,3) ≡ (2)1	1	−1	1	−1	i	−i	i	−i	1	1	−1	−1	−i	i	i	−i
(2,3) ≡(2)2	1	−1	1	−1	i	−i	i	−i	−1	−1	1	1	i	−i	−i	i
(2,1) ≡ (2)3	1	−1	1	−1	−i	i	−i	i	−1	−1	1	1	−i	i	i	−i
(3,3) ≡ (3)0	1	−1	−1	1	i	−i	−i	i	−i	i	−i	i	1	1	−1	−1
(1,1) ≡ (3)1	1	−1	−1	1	−i	i	i	−i	i	−i	i	−i	1	1	−1	−1
(1,3) ≡(3)2	1	−1	−1	1	i	−i	−i	i	i	−i	i	−i	−1	−1	1	1
(3,1) ≡ (3)3	1	−1	−1	1	−i	i	i	−i	−i	i	−i	i	−1	−1	1	1

Table AIII. Addition of Class Indices $\mathbb{Z}_2 \times \mathbb{Z}_2$

†	(0)	(1)	(2)	(3)
(0)	(0)	(1)	(2)	(3)
(1)	(1)	(0)	(3)	(2)
(2)	(2)	(3)	(0)	(1)
(3)	(3)	(2)	(1)	(0)

In Tables AI and AII, we have divided the 16 group elements of $\mathbb{Z}_4 \times \mathbb{Z}_4$ into four classes. The class (0) contains the indices associated to the Poincaré generators $\{(0)_0, (0)_1, (0)_2, (0)_3\}$. The class (0) is itself a group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The classes (1), (2), and (3) contain indices which remain in the same class when you add elements of the class (0). This indicates that the classes (0), (1), (2), (3) provide the cosets of indices available for building novel invariant multiplets of generators.

From Table AI, we obtain Table AIII for the addition of class indices $(\mathbb{Z}_4 \times \mathbb{Z}_4)/(\mathbb{Z}_2 \times \mathbb{Z}_2) \approx \mathbb{Z}_2 \times \mathbb{Z}_2$. We write $(i \dagger j)$ instead of $(i) \dagger (j)$.

We write the Poincaré algebra in terms of irreps of the Lorentz subalgebra. For that the Lorentz generators $M^{\mu\nu}$ themselves are rewritten as

$$J_i \equiv \frac{1}{2}\epsilon_{ijk} M^{jk}, \qquad \hat{J}_i \equiv M^{0i},$$

$$T_{(0)i} \equiv \frac{1}{2}(J_i + \mathrm{i}\hat{J}_i), \qquad \bar{T}_{(0)i} \equiv \frac{1}{2}(J_i - \mathrm{i}\hat{J}_i).$$

The Poincaré algebra becomes

$$[\![T_{(0)i}, T_{(0)j}]\!] = \mathrm{i}\epsilon_{ijk} T_{(0)k}, \; [\![T_{(0)i}, \bar{T}_{(0)j}]\!] = 0, \; [\![\bar{T}_{(0)i}, \bar{T}_{(0)j}]\!] = \mathrm{i}\epsilon_{ijk} \bar{T}_{(0)k}.$$
$$[\![T_{(0)i}, P_{(0)\nu}]\!] = -\mathrm{i}\sigma^P(i,0)_\nu{}^\rho \, P_{(0)\rho}, \; [\![\bar{T}_{(0)i}, P_{(0)\nu}]\!] = -\mathrm{i}q_{(0)i,(0)\nu} \, P_{(0)\rho} \bar{\sigma}^P(i,0)^\rho{}_\nu,$$
$$[\![P_{(0)\mu}, P_{(0)\nu}]\!] = 0, \tag{A1}$$

for which the q-commutators coincide with commutators. This algebra is extended for $f, i, j, k, l = 1, 2, 3$ through the *minimal vector clover extension*:

$$[\![T_{(0)i}, P_{(f)\nu}]\!] = -\mathrm{i}\sigma^P(i, f)_\nu{}^\rho \, P_{(f)\rho},$$
$$[\![\bar{T}_{(0)i}, P_{(f)\nu}]\!] = -\mathrm{i}q_{(0)i,(f)\nu} \, P_{(f)\rho} \, \bar{\sigma}^P(i, f)^\rho{}_\nu, \tag{A2}$$

$$[\![T_{(0)i}, T_{(f)s}]\!] = -\mathrm{i}\sigma^T(i, f)_s{}^t \, T_{(f)t}, \qquad [\![\bar{T}_{(0)i}, T_{(f)s}]\!] = 0,$$
$$[\![T_{(0)i}, \bar{T}_{(f)\dot{s}}]\!] = 0, \qquad [\![\bar{T}_{(0)i}, \bar{T}_{(f)\dot{s}}]\!] = -\mathrm{i}q_{(0)i,(f)s} \bar{T}_{(f)\dot{t}} \, \bar{\sigma}^T(i, f)^{\dot{t}}{}_{\dot{s}}, \tag{A3}$$

$$[\![P_{(0)\mu}, P_{(f)\nu}]\!] = 0, \qquad [\![P_{(f)\mu}, P_{(f)\nu}]\!] = 0, \tag{A4}$$

$$[\![P_{(0)\mu}, T_{(f)t}]\!] = 0, \qquad [\![P_{(0)\mu}, \bar{T}_{(f)\dot{s}}]\!] = 0, \tag{A5}$$

$$[\![T_{(i)s}, T_{(j)t}]\!] = 0, \qquad [\![T_{(i)s}, \bar{T}_{(j)\dot{t}}]\!] = 0, \qquad [\![\bar{T}_{(i)\dot{s}}, \bar{T}_{(j)\dot{t}}]\!] = 0, \tag{A6}$$

Table AIV. σ^T Matrices

(f)	Spin(1, 0)	$\sigma^T(1, f)$	$\sigma^T(2, f)$	$\sigma^T(3, f)$
(0)	$T_{(0)}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
(1)	$T_{(1)}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
(2)	$T_{(2)}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$
(3)	$T_{(3)}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$[\![P_{(i)\mu}, P_{(j)\nu}]\!] = \eta^r(i, j)_{\mu\nu} T_{(i\dagger j)r} + \hat{\eta}^{\dot{r}}(i, j)_{\mu\nu} \overline{T}_{(i\dagger j)\dot{r}}; \quad i \neq j, \tag{A7}$$

$$[\![T_{(k)r}, P_{(l)\mu}]\!] = \delta_{kl} K_r(l)_\mu^{\ \nu} P_{(0)\nu},$$

$$[\![\overline{T}_{(k)\dot{r}}, P_{(l)\mu}]\!] = \delta_{kl} \hat{K}_{\dot{r}}(l)_\mu^{\ \nu} P_{(0)\nu}, \tag{A8}$$

An adequate choice for the σ^T-matrices for spin (1, 0) irreps in the different classes is given in Table AIV with $\overline{\sigma}^T(j, f) = \sigma^T(j, f)^{*tr}$. A suited choice of spin $(\frac{1}{2}, \frac{1}{2})$ irreps for the different classes is given in Table AV for generic spin $(\frac{1}{2}, \frac{1}{2})$ 4-vectors $P_{(f)}$ with $\overline{\sigma}^P(j, f) = \sigma^P(j, f)^{*tr}$.

Table AV. σ^P Matrices

(f)	Spin$(\frac{1}{2}, \frac{1}{2})$	$\sigma^P(1, f)$	$\sigma^P(2, f)$	$\sigma^P(3, f)$
(0)	$P_{(0)}$	$\frac{1}{2}\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$
(1)	$P_{(1)}$	$\frac{1}{2}\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$
(2)	$P_{(2)}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$
(3)	$P_{(3)}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Table AVI. $\eta^r(i,j)$ Matrices, $ijk \in \{123, 231, 312\}$

$\eta^1(i,j)$	$\eta^2(i,j)$	$\eta^3(i,j)$
$a_k \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$a_k \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$a_k \begin{bmatrix} 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

The η-matrices for $ijk \in \{123, 231, 312\}$ are given in Table AVI, and the further η-matrices are obtained using (A9).

$$-q_{(j)\nu,(i)\mu}\eta^r(i,j)_{\mu\nu} = \eta^r(j,i)_{\nu\mu} = \hat{\eta}^{\dot{r}}(i,j)^*_{\mu\nu}, \tag{A9}$$

$$-q_{(j)\nu,(i)\mu}\hat{\eta}^{\dot{r}}(i,j)_{\mu\nu} = \hat{\eta}^{\dot{r}}(j,i)_{\nu\mu} = \eta^r(i,j)^*_{\mu\nu}.$$

The K-matrices are given in Table AVII. The $\hat{K}$-matrices are obtained using (A10).

$$\hat{K}_{\dot{r}}(l)_\mu^{\ \nu} = -q_{(l)r,(l)\mu}K_r(l)_\mu^{\ \nu*}. \tag{A10}$$

The η- and the K-matrices are constrained by the conditions (A11)–(A12).

$$a_1b_1 + a_2b_2 + a_3b_3 = \frac{3}{2}r_0(1-\mathrm{i}); \quad r_0 \in \mathbb{R}, \tag{A11}$$

$$a_jb_j - \mathrm{i}a_j^*b_j^* = r_0(1-\mathrm{i}); \quad j = 1, 2, 3. \tag{A12}$$

A particular choice of these constraints respecting the symmetry among the novel classes is given by (A13):

$$a_j = a; \quad b_j = b(1-\mathrm{i});\ a, b \in \mathbb{R}\backslash\{0\}; \quad r_0 = 2ab;\ j = 1, 2, 3. \tag{A13}$$

Table AVII. $K_r(l)$ Matrices

(f)	$K_1(f)$	$K_2(f)$	$K_3(f)$
(1)	$b_1 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$b_1 \begin{bmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}$	$b_1 \begin{bmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 1 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$
(2)	$b_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$b_2 \begin{bmatrix} 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}$	$b_2 \begin{bmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 1 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$
(3)	$b_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$b_3 \begin{bmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \mathrm{i} & 0 & 0 & 0 \end{bmatrix}$	$b_3 \begin{bmatrix} 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

The existence and exhibition of a differential representation for the clover extension is presented in the main text.

APPENDIX B: METRICS AND USEFUL IDENTITIES

The metrics $\epsilon^T(f)$ compatible with the spin (1, 0) irreps $T_{(f)}$ and the metrics $\overline{\epsilon^{\bar{T}}}(f)$ compatible with the spin (0, 1) irreps $\bar{T}_{(f)}$ are given by

$$\epsilon^T(0)_{rs} = \epsilon^T(0)^{rs} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{rs}, \epsilon^T(i)_{rs} = \epsilon^T(i)^{rs} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{rs}, \tag{B1}$$

$$\overline{\epsilon^{\bar{T}}}(0)_{\dot{r}\dot{s}} = \overline{\epsilon^{\bar{T}}}(0)^{\dot{r}\dot{s}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{\dot{r}\dot{s}}, \overline{\epsilon^{\bar{T}}}(i)_{\dot{r}\dot{s}} = \overline{\epsilon^{\bar{T}}}(i)^{\dot{r}\dot{s}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{\dot{r}\dot{s}}, \tag{B2}$$

for $i = 1, 2, 3$. The metrics $\epsilon^P(f)$ compatible with the spin $(\frac{1}{2}, \frac{1}{2})$ irreps $P_{(f)}$ are given by

$$\epsilon^P(0)_{\mu\nu} = \epsilon^P(0)^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{\mu\nu}, \tag{B3}$$

$$\epsilon^P(i)_{\mu\nu} = \epsilon^P(i)^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{\mu\nu}, \tag{B4}$$

for $i = 1, 2, 3$.

Some useful identities among the structure constant arrays and the metrics are given along the lines of an analogous appendix in Piguet and Sibold (1986):

$$[\mathrm{i}\sigma^a(j, f), \mathrm{i}\sigma^a(k, f)] = \mathrm{i}\epsilon_{jkl}(\mathrm{i}\sigma^a(l, f)), \tag{B5}$$

$$\left[-\mathrm{i}q_{(0)j,(f)}\overline{\sigma^{\bar{a}}}(j, f), -\mathrm{i}q_{(0)k,(f)}\overline{\sigma^{\bar{a}}}(k, f)\right] = \mathrm{i}\epsilon_{jkl}\left(-\mathrm{i}q_{(0)l,(f)}\overline{\sigma^{\bar{a}}}(l, f)\right), \tag{B6}$$

where the commutator has the usual meaning, and a stands for any arbitrary irrep.

$$\hat{K}^*_r(i)_\sigma{}^\mu \epsilon^T(i)^{rs} \epsilon^P(i)^{\sigma\nu} K_s(i)_\nu{}^\rho = 3b_i^2 \epsilon^P(0)^{\mu\rho}, \tag{B7}$$

$$K^*_{\dot{r}}(i)_\sigma{}^\mu \overline{\epsilon^{\bar{T}}}(i)^{\dot{r}\dot{s}} \epsilon^P(i)^{\sigma\nu} \hat{K}_{\dot{s}}(i)_\nu{}^\rho = 3b_i^{*2} \epsilon^P(0)^{\mu\rho}, \tag{B8}$$

$$q_{(i)\mu,(j)\sigma}\eta^{r}(i,j)_{\mu\sigma}\epsilon^{P}(i)^{\mu\alpha}\eta^{s}(i,j)_{\alpha\beta}\epsilon^{P}(j)^{\sigma\beta} = 4\,\mathrm{i}\epsilon^{T}(i\dagger j)^{rs}a^{2}_{i\dagger j}, \tag{B9}$$

$$q_{(i)\mu,(j)\sigma}\hat{\eta}^{\dot r}(i,j)_{\mu\sigma}\epsilon^{P}(i)^{\mu\alpha}\hat{\eta}^{\dot s}(i,j)_{\alpha\beta}\epsilon^{P}(j)^{\sigma\beta} = -4\,\mathrm{i}\overline{\epsilon^{\bar T}}(i\dagger j)^{\dot r\dot s}a^{*2}_{i\dagger j}, \tag{B10}$$

$$q_{(i)\mu,(j)\sigma}\eta^{r}(i,j)_{\mu\sigma}\epsilon^{P}(i)^{\mu\alpha}\hat{\eta}^{\dot s}(i,j)_{\alpha\beta}\epsilon^{P}(j)^{\sigma\beta} = 0, \tag{B11}$$

$$q_{(i)\mu,(j)\sigma}\hat{\eta}^{\dot r}(i,j)_{\mu\sigma}\epsilon^{P}(i)^{\mu\alpha}\eta^{s}(i,j)_{\alpha\beta}\epsilon^{P}(j)^{\sigma\beta} = 0, \tag{B12}$$

$$\sigma^{P}(l,i)_{\mu}{}^{\rho}\hat{\eta}^{\dot n}(i,j)_{\rho\nu} + (2\delta_{li}-1)\hat{\eta}^{\dot n}(i,j)_{\mu\rho}\sigma^{P}(l,j)_{\nu}{}^{\rho} = 0, \tag{B13}$$

$$\sigma^{P}(l,i)_{\mu}{}^{\rho}\eta^{n}(i,j)_{\rho\nu} + (2\delta_{li}-1)\eta^{n}(i,j)_{\mu\rho}\sigma^{P}(l,j)_{\nu}{}^{\rho} = \eta^{r}(i,j)_{\mu\nu}\sigma^{T}(l,i\dagger j)_{r}{}^{n},$$

$$\bar\sigma^{P}(l,i)_{\mu}{}^{\rho}\eta^{n}(i,j)_{\rho\nu} + (2\delta_{lj}-1)\eta^{n}(i,j)_{\mu\rho}\bar\sigma^{P}(l,j)_{\nu}{}^{\rho} = 0, \tag{B14}$$

$$\bar\sigma^{P}(l,j)_{\nu}{}^{\rho}\hat{\eta}^{\dot n}(i,j)_{\mu\rho} + (2\delta_{lj}-1)\hat{\eta}^{\dot n}(i,j)_{\rho\nu}\bar\sigma^{P}(l,i)_{\mu}{}^{\rho} = \hat{\eta}^{\dot r}(i,j)_{\mu\nu}\bar\sigma^{\bar T}(l,i\dagger j)_{\dot r}{}^{\dot n},$$

$$K_{j}(k)_{\mu}{}^{\nu}\bar\sigma^{P}(l,0)_{\nu}{}^{\rho} = \bar\sigma^{P}(l,k)_{\mu}{}^{\nu}K_{j}(k)_{\nu}{}^{\rho}, \tag{B15}$$

$$K_{j}(k)_{\mu}{}^{\nu}\sigma^{P}(l,0)_{\nu}{}^{\rho} = (2\delta_{lk}-1)\sigma^{P}(l,k)_{\mu}{}^{\nu}K_{j}(k)_{\nu}{}^{\rho} + \sigma^{T}(l,k)_{j}{}^{m}K_{m}(k)_{\mu}{}^{\rho}. \tag{B16}$$

$$\begin{aligned}&\left\{\eta^{r}(1,2)_{\rho\mu}K_{r}(3)_{\nu}{}^{\sigma} + \hat{\eta}^{\dot r}(1,2)_{\rho\mu}\hat{K}_{\dot r}(3)_{\nu}{}^{\sigma}\right\}\\ &\quad + q_{(1)\rho,(2)\mu+(3)\nu}\left\{\eta^{r}(2,3)_{\mu\nu}K_{r}(1)_{\rho}{}^{\sigma} + \hat{\eta}^{\dot r}(2,3)_{\mu\nu}\hat{K}_{\dot r}(1)_{\rho}{}^{\sigma}\right\}\\ &\quad + q_{(1)\rho+(2)\mu,(3)\nu}\left\{\eta^{r}(3,1)_{\nu\rho}K_{r}(2)_{\mu}{}^{\sigma} + \hat{\eta}^{\dot r}(3,1)_{\nu\rho}\hat{K}_{\dot r}(2)_{\mu}{}^{\sigma}\right\} = 0.\end{aligned} \tag{B17}$$

APPENDIX C: ALGEBRAIC RELATIONS AMONG COVARIANT DERIVATIVES

We list now some algebraic relations involving covariant derivatives. As well as with the relations among the coupling constant arrays, we identify the correspondence to analogous relations in the supersymmetric case in Piguet and Sibold (1986). These relations will play a crucial role in the construction of QFT models.

$$\left[\!\left[D_{\bar T_{(i)\dot r}}, D_{P^{\sigma}_{(j)}}D_{P_{(j)\sigma}}\right]\!\right] = 2\,\mathrm{i}\delta_{ij}\hat{K}_{\dot r}(j)_{\sigma}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}D_{P^{\sigma}_{(j)}}, \tag{C1}$$

$$\left[\!\left[D_{T_{(i)r}}, D_{P^{\sigma}_{(j)}}D_{P_{(j)\sigma}}\right]\!\right] = 2\,\mathrm{i}\delta_{ij}K_{r}(j)_{\sigma}{}^{\mu}\partial_{\chi_{(0)}^{-\mu}}D_{P^{\sigma}_{(j)}}, \tag{C2}$$

$$\left[\!\left[D_{P_{(j)\sigma}}, D_{T^{r}_{(i)}}D_{T_{(i)r}}\right]\!\right] = -2\,\mathrm{i}\delta_{ij}K_{r}(j)_{\sigma}{}^{\mu}D_{T^{r}_{(i)}}\partial_{\chi_{(0)}^{-\mu}}, \tag{C3}$$

$$\left[\!\left[D_{P_{(j)\sigma}}, D_{\bar T^{\dot r}_{(i)}}D_{\bar T_{(i)\dot r}}\right]\!\right] = -2\,\mathrm{i}\delta_{ij}\hat{K}_{\dot r}(j)_{\sigma}{}^{\mu}D_{\bar T^{\dot r}_{(i)}}\partial_{\chi_{(0)}^{-\mu}}, \tag{C4}$$

$$\begin{aligned}\big[\!\big[D_{P_{(j)\mu}}, D_{P^{\sigma}_{(i)}} D_{P_{(i)\sigma}} \big]\!\big] &= 2\,\mathrm{i}(\eta^{r}(j,i)_{\mu\sigma} D_{T_{(i\dagger j)r}} + \hat{\eta}^{\dot r}(j,i)_{\mu\sigma} D_{\bar{T}_{(j\dagger i)\dot r}}) D_{P^{\sigma}_{(i)}} \\ &= 2\big[\!\big[D_{P_{(j)\mu}}, D_{P_{(i)\sigma}} \big]\!\big] D_{P^{\sigma}_{(i)}}, \quad i \neq j, \end{aligned} \tag{C5}$$

$$\begin{aligned} D_{P^{\sigma}_{(j)}} D_{T_{(i)r}} D_{P_{(j)\sigma}} &= \frac{1}{2} q_{(i)r,(j)\sigma} \big(D_{P^{\sigma}_{(j)}} D_{P_{(j)\sigma}} \big) D_{T_{(i)r}} \\ &\quad + \frac{1}{2} q_{(j)\sigma,(i)r} D_{T_{(i)r}} \big(D_{P^{\sigma}_{(j)}} D_{P_{(j)\sigma}} \big), \end{aligned} \tag{C6}$$

$$\begin{aligned} D_{T^{r}_{(i)}} D_{P_{(j)\sigma}} D_{T_{(i)r}} &= \frac{1}{2} q_{(i)r,(j)\sigma} D_{P_{(j)\sigma}} \big(D_{T^{r}_{(i)}} D_{T_{(i)r}} \big) \\ &\quad + \frac{1}{2} q_{(j)\sigma,(i)r} \big(D_{T^{r}_{(i)}} D_{T_{(i)r}} \big) D_{P_{(j)\sigma}}, \end{aligned} \tag{C7}$$

$$\begin{aligned} D_{\bar{T}^{\dot r}_{(i)}} D_{P_{(j)\sigma}} D_{\bar{T}_{(i)\dot r}} &= \frac{1}{2} q_{(i)r,(j)\sigma} D_{P_{(j)\sigma}} \big(D_{\bar{T}^{\dot r}_{(i)}} D_{\bar{T}_{(i)\dot r}} \big) \\ &\quad + \frac{1}{2} q_{(j)\sigma,(i)r} \big(D_{\bar{T}^{\dot r}_{(i)}} D_{\bar{T}_{(i)\dot r}} \big) D_{P_{(j)\sigma}}, \end{aligned} \tag{C8}$$

$$\begin{aligned} D_{P^{\sigma}_{(i)}} D_{P_{(j)\mu}} D_{P_{(i)\sigma}} &= \frac{1}{2} q_{(j)\mu,(i)\sigma} \big(D_{P^{\sigma}_{(i)}} D_{P_{(i)\sigma}} \big) D_{P_{(j)\mu}} \\ &\quad + \frac{1}{2} q_{(i)\sigma,(j)\mu} D_{P_{(j)\mu}} \big(D_{P^{\sigma}_{(i)}} D_{P_{(i)\sigma}} \big), \end{aligned} \tag{C9}$$

$$\begin{aligned} D_{T^{s}_{(i)}} D_{\overset{(-)}{T}_{(j)r}} D_{T_{(i)s}} &= \frac{1}{2} q_{(i)s,(j)r} D_{\overset{(-)}{T}_{(j)r}} \big(D_{T^{s}_{(i)}} D_{T_{(i)s}} \big) \\ &\quad + \frac{1}{2} q_{(j)r,(i)s} \big(D_{T^{s}_{(i)}} D_{T_{(i)s}} \big) D_{\overset{(-)}{T}_{(j)r}}, \end{aligned} \tag{C10}$$

$$\begin{aligned} D_{\bar{T}^{\dot s}_{(i)}} D_{\overset{(-)}{T}_{(j)r}} D_{\bar{T}_{(i)\dot s}} &= \frac{1}{2} q_{(i)s,(j)r} D_{\overset{(-)}{T}_{(j)r}} \big(D_{\bar{T}^{\dot s}_{(i)}} D_{\bar{T}_{(i)\dot s}} \big) \\ &\quad + \frac{1}{2} q_{(j)r,(i)s} \big(D_{\bar{T}^{\dot s}_{(i)}} D_{\bar{T}_{(i)\dot s}} \big) D_{\overset{(-)}{T}_{(j)r}} \end{aligned} \tag{C11}$$

$$\begin{aligned} \big[\!\big[D_{T^{r}_{(i)}} D_{T_{(i)r}}, D_{P^{\sigma}_{(j)}} D_{P_{(j)\sigma}} \big]\!\big] &= \delta_{ij} \big(-6\, b_i^2 \Box - 4\,\mathrm{i} \hat{K}^{*}_{r}(i)_{\sigma}{}^{\mu} D_{T^{r}_{(i)}} D_{P^{\sigma}_{(i)}} \partial_{\chi^{-\mu}_{(0)}} \big) \\ &= \delta_{ij} \big(6\, b_i^2 \Box + 4\,\mathrm{i} K_{r}(i)_{\sigma}{}^{\mu} D_{P^{\sigma}_{(i)}} D_{T^{r}_{(i)}} \partial_{\chi^{-\mu}_{(0)}} \big), \end{aligned} \tag{C12}$$

$$\begin{aligned} \big[\!\big[D_{\bar{T}^{\dot r}_{(i)}} D_{\bar{T}_{(i)\dot r}}, D_{P^{\sigma}_{(j)}} D_{P_{(j)\sigma}} \big]\!\big] &= \delta_{ij} \big(-6 \big(b_i^{*}\big)^2 \Box - 4\,\mathrm{i} K^{*}_{\dot r}(i)_{\sigma}{}^{\mu} D_{\bar{T}^{\dot r}_{(i)}} D_{P^{\sigma}_{(i)}} \partial_{\chi^{-\mu}_{(0)}} \big) \\ &= \delta_{ij} \big(6 \big(b_i^{*}\big)^2 \Box + 4\,\mathrm{i} \hat{K}_{\dot r}(i)_{\sigma}{}^{\mu} D_{P^{\sigma}_{(i)}} D_{\bar{T}^{\dot r}_{(i)}} \partial_{\chi^{-\mu}_{(0)}} \big), \end{aligned} \tag{C13}$$

$$\left[\!\left[D_{P^{\mu}_{(i)}} D_{P_{(i)\mu}}, D_{P^{\sigma}_{(j)}} D_{P_{(j)\sigma}} \right]\!\right] = 4\,\mathrm{i}\, q_{(i)\mu,(j)\sigma}\, \eta^{r}(i,\ j)_{\mu\sigma}\, D_{T_{(i\dagger j)r}} D_{P^{\mu}_{(i)}} D_{P^{\sigma}_{(j)}} + \\ + 4\,\mathrm{i}\, q_{(i)\mu,(j)\sigma}\, \hat{\eta}^{\dot{r}}(i,\ j)_{\mu\sigma}\, D_{\bar{T}_{(i\dagger j)\dot{r}}} D_{P^{\mu}_{(i)}} D_{P^{\sigma}_{(j)}} \\ + 8\,\mathrm{i}(a_{(i\dagger j)})^{2} D_{T^{r}_{(i\dagger j)}} D_{T_{(i\dagger j)r}} \\ - 8\,\mathrm{i}\left(a^{*}_{(i\,\dagger\,j)}\right)^{2} D_{\bar{T}^{\dot{r}}_{(i\dagger j)}} D_{\bar{T}_{(i\dagger j)\dot{r}}}, \tag{C14}$$

$$D_{P^{\sigma}_{(i)}} D_{\overset{(-)}{T}{}^{r}_{(i)}} D_{\overset{(-)}{T}_{(i)r}} D_{P_{(i)\sigma}} = D_{\overset{(-)}{T}{}^{r}_{(i)}} D_{P^{\sigma}_{(i)}} D_{P_{(i)\sigma}} D_{\overset{(-)}{T}_{(i)r}}, \tag{C15}$$

$$D_{P^{\sigma}_{(i)}} D_{\overset{(-)}{T}{}^{r}_{(i)}} D_{\overset{(-)}{T}_{(i)r}} D_{P_{(i)\sigma}} - \frac{1}{2} D_{P^{\sigma}_{(i)}} D_{P_{(i)\sigma}} D_{\overset{(-)}{T}{}^{r}_{(i)}} D_{\overset{(-)}{T}_{(i)r}} \\ - \frac{1}{2} D_{\overset{(-)}{T}{}^{r}_{(i)}} D_{\overset{(-)}{T}_{(i)r}} D_{P^{\sigma}_{(i)}} D_{P_{(i)\sigma}} = -3 b_{i}^{(-)2} \Box. \tag{C16}$$

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